The Virial Theorem

The (gravitational) Virial Theorem relates the gravitational potential energy of a system to the kinetic energy and provides an insight into the stability of the system.

Its derivation is simple, if somewhat tedious.

For a collection of $N$ point masses the scalar moment of inertia about the origin is given by

$$I = \sum_{k=1}^{n} m_k r_k^2 = \sum_{k=1}^{n} m_k r_k^2$$

where $m_k$ and $r_k$ are the mass and position of the $k^{th}$ particle.

We can define a the scalar virial $G$ as the time derivative of $I$

$$2G = \frac{dI}{dt} = 2 \sum_{k=1}^{n} m_k \frac{dr_k}{dt} \cdot r_k = 2 \sum_{k=1}^{n} p_k \cdot r_k$$

and we can write the time derivative of the virial as

$$\frac{dG}{dt} = \sum_{k=1}^{n} p_k \cdot \frac{dr_k}{dt} + \sum_{k=1}^{n} \frac{dp_k}{dt} \cdot r_k$$

so

$$\frac{dG}{dt} = \sum_{k=1}^{n} m_k \frac{dr_k}{dt} \cdot \frac{dr_k}{dt} + \sum_{k=1}^{n} F_k \cdot r_k$$

as $\frac{dp_k}{dt} = F_k$ is the net force on particle $k$.

We can re-write the first term as

$$\sum_{k=1}^{n} m_k \frac{dr_k}{dt} \cdot \frac{dr_k}{dt} = \sum_{k=1}^{n} m_k v_k^2 = 2T$$

where $T$ is the kinetic energy of the system.

This is as far as I would ask you to go in an exam as far as a derivation is concerned.

The second term is somewhat more complicated. You can probably see that with the force on $k$ from all the other particles $F_k$ involved that this might have something to do with the gravitational potential energy. It turns-out (see http://en.wikipedia.org/wiki/Virial_theorem or http://www.math.ucr.edu/home/baez/virial.html for a full derivation - not too difficult, just a little tedious) that the second term is

$$\sum_{k=1}^{n} F_k \cdot r_k = \Omega$$

where $\Omega$ is the gravitational potential energy.

So,

$$\frac{dG}{dt} = 2T + \Omega$$

The time average of $\frac{dG}{dt}$ is

$$\langle \frac{dG}{dt} \rangle_t = \frac{(G(t) - G(0))}{t}$$
as \( t \) tends to infinity. The virial theorem applies to *bound systems*, that is there is a maximum \( G_{\text{max}} \) and minimum \( G_{\text{min}} \) value of \( G \) which the system can have. So as \( t \) tends to infinity

\[
\lim_{t \to \infty} \left\langle \frac{dG}{dt} \right\rangle = \lim_{t \to \infty} \left| \frac{G(t) - G(0)}{t} \right| \leq \lim_{t \to \infty} \frac{G_{\text{max}} - G_{\text{min}}}{t} = 0
\]

Therefore the full form of the gravitational virial theorem (virial equilibrium) is

\[
0 = 2T + \Omega
\]

and a gravitational system will tend to a state in which \(-\Omega = 2T\). If fact, gravitational systems tend to do this very rapidly, a bound system that is not in virial equilibrium will change its configuration very rapidly to get into virial equilibrium. Therefore we expect bound systems to be close to virial equilibrium when we observe them. The state of a system is often quoted in terms of the virial ratio

\[
Q_{\text{vir}} = \frac{T}{-\Omega}
\]

where a system in virial equilibrium has \( Q_{\text{vir}} = 0.5 \). If the system has \( Q > 1 \) it is unbound (and the virial theorem doesn’t apply except to tell us it is unbound).

The generic form of the potential energy is

\[
\Omega = -\alpha \frac{GM^2}{R}
\]

where \( \alpha \) is a structure parameter of order unity, and \( R \) is some characteristic size of the system (usually the core or half-mass radius). The kinetic energy is

\[
T = \frac{1}{2} M \sigma^2 = \frac{3}{2} M c_s^2
\]

where \( \sigma \) is the 3D velocity dispersion and \( c_s \) is the (1D) sound speed. Therefore the virial ratio can be written

\[
Q = \frac{1 \sigma^2 R}{2 \alpha G M}
\]

Very often the mass of a system is not known. If we assume that the system is in virial equilibrium \((Q = 0.5)\) then we can calculate a *virial mass*

\[
M_{\text{vir}} = \frac{1}{\alpha G} \sigma^2 R
\]

If the virial mass is greater than the true mass, then the system has \( Q < 0.5 \) and is stable and collapsing. If the virial mass is less than the true mass then the system has \( Q > 0.5 \) and is expanding.

An interesting consequence of the virial theorem is that gravitational systems have negative specific heat. That is, if you remove energy, they get hotter. This can be seen quite simply, the total energy is

\[
E_{\text{tot}} = T + \Omega
\]

where \( E_{\text{tot}} < 0 \) for a bound system and the virial theorem tells us that

\[
2T + \Omega = 0
\]
therefore

\[ E_{\text{tot}} = \frac{\Omega}{2} \]

If we remove energy \( E_{\text{tot}} \) becomes more negative, and so (once the system has re-virialised) \( \Omega \) must be more negative, therefore \( T \) must become more positive - the system heats-up. And the more energy is removed from the system, the hotter it becomes.