The surface temperature of a planet

Note that this derivation is for a rapidly spinning planet which can be assumed to absorb and radiate evenly over the whole surface area.

The power $W$ ($\text{J s}^{-1}$) radiated per unit area by a blackbody is

$$W = \sigma T^4$$

with $T$ in K, and $\sigma$ is the Stefan-Boltzmann constant ($5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$). Therefore the energy $E$ radiated per unit time from a spherical blackbody or radius $r$ is

$$E = 4\pi r^2 W$$

and the temperature of that blackbody is

$$T = \left(\frac{E}{4\pi\sigma r^2}\right)^{1/4}$$

The energy from a star of luminosity $L$ at a distance $d$ incident on a planet of radius $r$ is

$$E_{\text{star-planet}} = \frac{L \pi r^2}{4\pi d^2}$$

However, not all of the stellar energy that is incident with a planet reaches the planet’s surface.

A fraction $\alpha$ (often $A$) of the energy is reflected directly back into space (mainly by clouds or ice). The fraction of energy reflected is the albedo of the planet. For example the albedo of Earth is $\sim 0.3$, and of Venus $\sim 0.7$ (the large albedo of Venus is due to its complete cloud cover).

In addition, a fraction $\beta$ of energy can be absorbed by the atmosphere (especially in the UV and IR) never reaching the ground (this is the reason for the temperature inversion at $\sim 10$ km at the top of the troposphere and the bottom of the stratosphere). We will ignore this factor, but see http://www.dangermouse.net/gurps/science/temps.html for a full derivation with this factor included.

The total energy that reaches the surface of a planet is

$$E_{\text{surf}} = (1 - \alpha)L \frac{\pi r^2}{4\pi d^2}$$

Solving for the surface temperature gives

$$T_{\text{surf}} = \left(\frac{(1 - \alpha)L}{16\pi\sigma d^2}\right)^{1/4}$$

Note that this is independent of the planet’s radius.

Putting-in values for the Earth and Sun where $L_{\odot} = 3.83 \times 10^{26}$ W, $d = 1.5 \times 10^{11}$ m and $\alpha = 0.31$ gives a $T_{\text{surf}} \sim 245$ K (note that if we’d done this including the effects of the atmospheric absorption of incoming radiation we would have got $\sim 250$ K).
But the average surface temperature of the Earth is $\sim 286$ K (13 C). This is because we have ignored the greenhouse effect - the trapping of heat radiated by the surface by the atmosphere (see your lecture notes and the PHY229 wiki). For the Earth the greenhouse effect increases the surface temperature by $35 - 40$ K!

Note that these are average surface temperatures: latitude, geography, seasons, local albedo, local cloud cover etc. cause variations in the earth’s surface temperature of $\pm 30 - 40$ K (compare Northern Canada with Arabia).

Another minor factor is heat released from inside a planet. The larger terrestrial planets have molten cores heated by radioactive decay. Heat is released in small amounts across the planet, but especially close to volcanoes, however this is normally a negligible amount. In extreme conditions - such as Io (and to a lesser extent Europa) - tidal forces from a massive nearby body can heat the interior of a planet/moon.

**Retaining an atmosphere**

The rms velocity of a gas particle of mass $m$ at a temperature $T$ is

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}}$$

where $k_B$ is Boltzmann’s constant ($1.38 \times 10^{-23}$ m$^2$ kg s$^{-2}$ K$^{-1}$). The escape velocity from a planet of radius $r$ and mass $M$ is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$$

where $G$ is Newton’s constant of gravitation ($6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$).

In order to retain an atmosphere for a significant time we require $v_{\text{rms}} < v_{\text{esc}}/6$.

The factor of $\sim 6$ comes from a Maxwellian velocity distribution, one particle in $10^{16}$ has more than $\sim 6$ times the rms velocity, and as high velocity molecules are lost, the Maxwellian re-establishes itself leading to more particles above the escape velocity.

For the Earth $v_{\text{esc}} = 11.2$ km s$^{-1}$. At a temperature of 286 K $v_{\text{rms}}$ for H$_2$ is 1.9 km s$^{-1}$, whilst for O$_2$ it is 0.5 km s$^{-1}$. Thus for hydrogen the high-velocity tail of the Maxwellian that is $> v_{\text{esc}}$ is sufficient for the loss of molecules into space.